# Some results on average-case matrix discrepancy

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# Discrepancy theory

“Divide a group of things into two similar groups”

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Cf. e.g. talks of Dan Spielman

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Set system $\mathcal{S} := \{S_1, \cdots, S_d\}$

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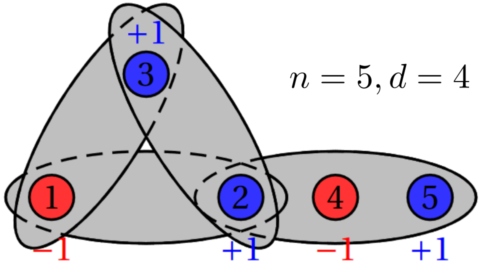
$$\mathrm{disc}(\chi) := \max_{i \in [d]} \left| \sum_{j \in S_i} \chi(j) \right|$$



$n = 5, d = 4$

$$\mathrm{disc}(\chi) = \max(0, 0, 1, 2) = 2$$

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**Goal**

Find a low-discrepancy coloring

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**Motivations / applications**

Combinatorics, computational geometry, experimental design, theory of approximation algorithms, …

Matousek '09 ; Chen&al '14 ; …

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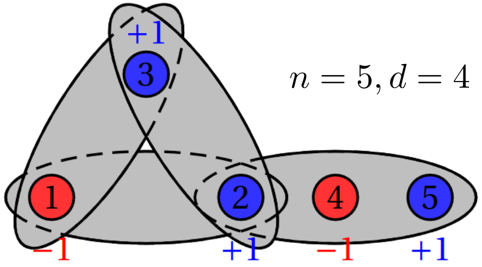

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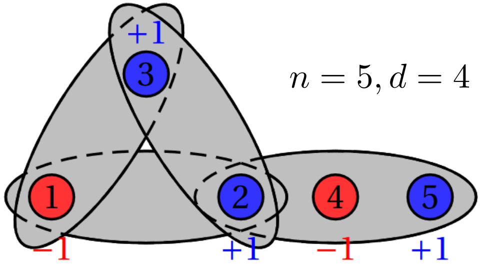

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## Discrepancy of a set system / a set of vectors

$$\mathrm{disc}(\mathcal{S}) = \min_{\varepsilon\in\{\pm1\}^n} \max_{i\in[d]} |\langle \varepsilon, a_i\rangle| = \min_{\varepsilon\in\{\pm1\}^n} \left\|\sum_{i=1}^n \varepsilon_i u_i\right\|_\infty$$

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Union bound

$$\mathbb{P}[\mathrm{disc}(\varepsilon) \leq \sqrt{2n\log(2d)}] > 0$$

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**Theorem** (Spencer ’85)

$$\boxed{\|u_i\|_\infty \leq 1 \;\Rightarrow\; \mathrm{disc}(u_1, \cdots, u_n) \leq 6\sqrt{n}}$$

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## Digression

**Komlós conjecture (70s)**

$\|u_i\|_2 \leq \sqrt{n}$ is sufficient for $\mathrm{disc}(\{u_i\}) \leq C\sqrt{n}$

Banaszczyk ‘98

Best-known result $\mathrm{disc}(\{u_i\}) \leq C\sqrt{n \log d}$

⚠ Hard result

~~Hoeffding’s inequality~~

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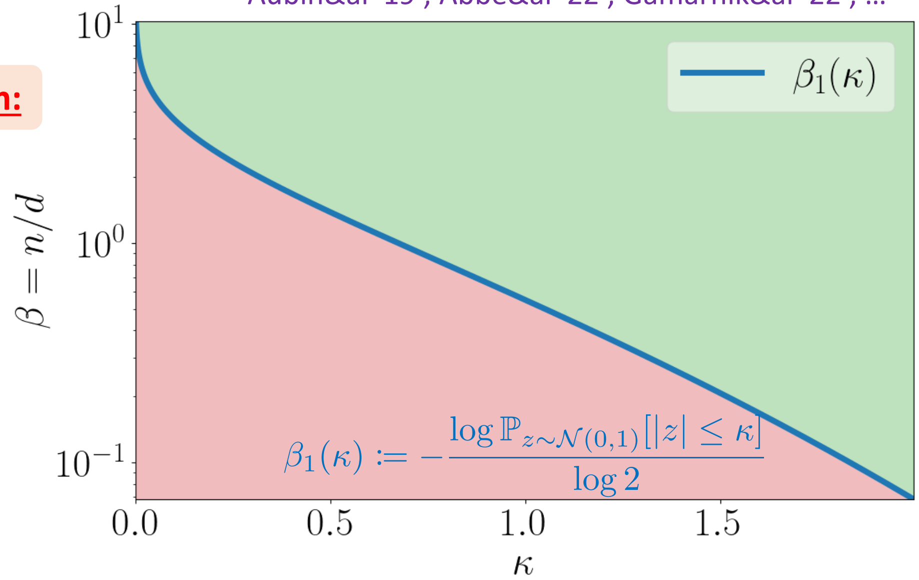

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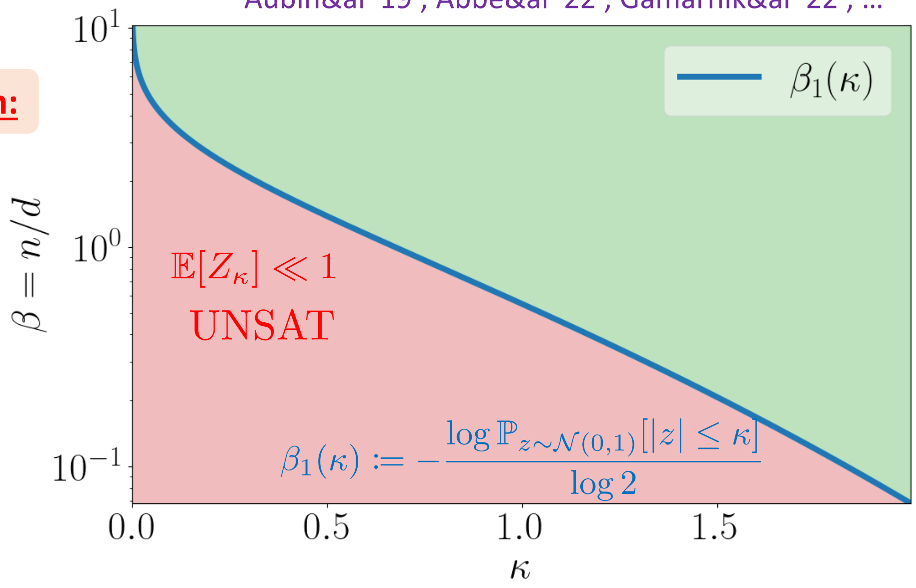

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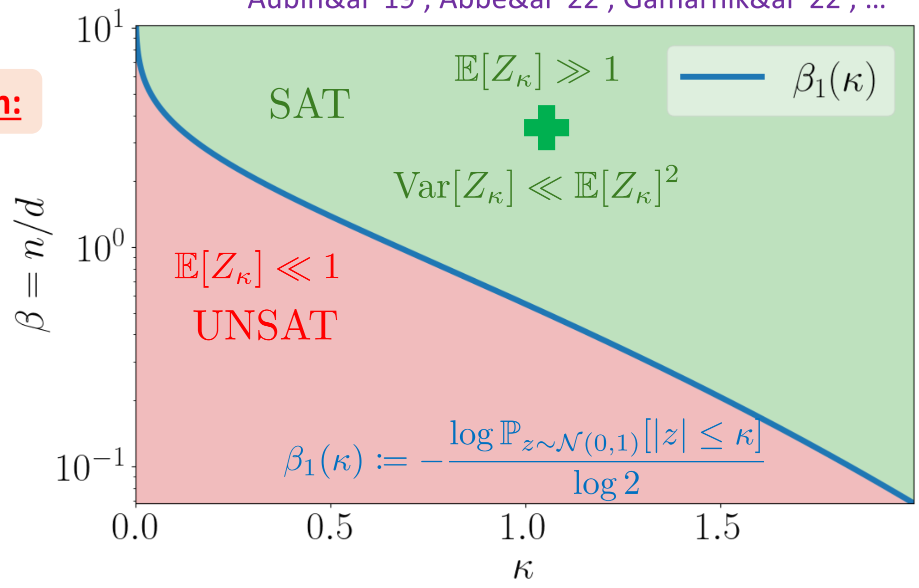

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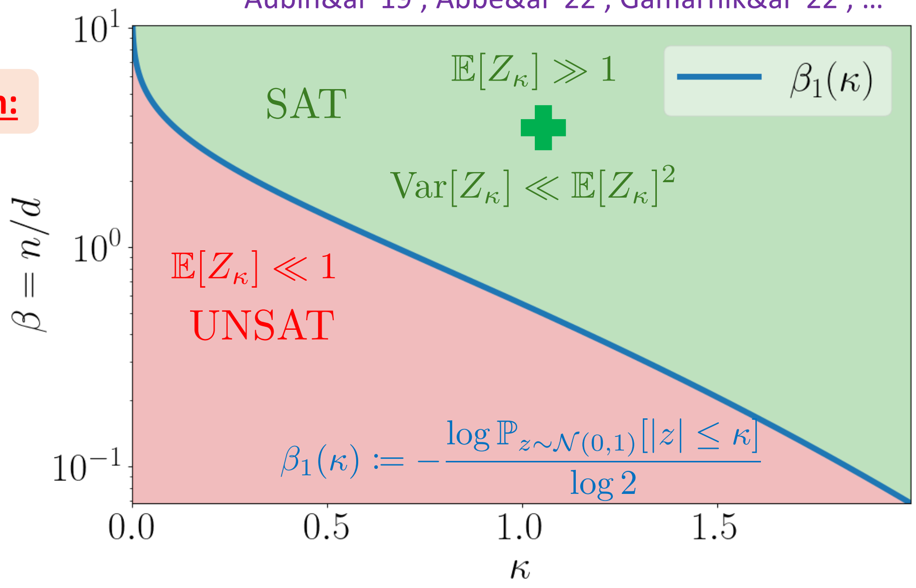

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- There is a **sharp satisfiability threshold.**
- The **annealed** free energy is correct: $Z_\kappa \simeq \mathbb{E}[Z_\kappa]$
- Much more **detailed properties** are known:

Structure of solution space, performance of solving algorithms, …

Barbier&al '24 ; El Alaoui & Gamarnik '24 ; …

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Applications in quantum random access codes, graph sparsification, …

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**Conjecture ("Matrix Spencer")**

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Linked to the "ammount of commutativity" of $\{\mathbf{A}_i\}$, see talks of A. Bandeira.

$\{\mathbf{A}_i\}$ commute ⇒ Spencer's theorem.

$\{\mathbf{A}_i\}$ "very non-commutative" ⇒ $\mathbb{E}_\varepsilon \| \sum_i \varepsilon_i \mathbf{A}_i \|_{\mathrm{op}} \lesssim \sqrt{n}$

Bandeira, Boedihardjo & van Handel '23

# Matrix discrepancy

$\|u_i\|_\infty \leq 1 \Rightarrow \mathrm{disc}(u_1, \cdots, u_n) \leq C\sqrt{n}$

What happens for **more complex discrepancy objectives** ?

Seminal example

$u_i \Rightarrow \mathbf{A}_i \in \mathbb{R}^{d \times d}$

$\| \cdot \|_\infty \Rightarrow \| \cdot \|_{\mathrm{op}}$ (Spectral norm)

Assume $\|\mathbf{A}_i\|_{\mathrm{op}} \leq 1$

**Random** signs $\varepsilon \sim \mathrm{Unif}(\{\pm 1\}^n)$

Non-commutative Khintchine inequality [Lust-Piquard & Pisier '91]

$$\Downarrow$$

$$\mathrm{disc}(\mathbf{A}_1, \cdots, \mathbf{A}_n) \leq C\sqrt{n \log d}$$

**Can we beat random signings ?**

**Matrix discrepancy**

$$\mathrm{disc}(\mathbf{A}_1, \cdots, \mathbf{A}_n) := \min_{\varepsilon \in \{\pm 1\}^n} \left\| \sum_{i=1}^n \varepsilon_i \mathbf{A}_i \right\|_{\mathrm{op}}$$

Applications in quantum random access codes, graph sparsification, …

Hopkins&al '22; Bansal&al '23; Batson&al '14; …

## Conjecture ("Matrix Spencer")

$$\boxed{\|\mathbf{A}_i\|_{\mathrm{op}} \leq 1 \Rightarrow \mathrm{disc}(\mathbf{A}_1, \cdots, \mathbf{A}_n) \leq C\sqrt{n}}$$

Zouzias '12; Meka'14

Linked to the "ammount of commutativity" of $\{\mathbf{A}_i\}$, see talks of A. Bandeira.

$\{\mathbf{A}_i\}$ commute $\Rightarrow$ Spencer's theorem.

$\{\mathbf{A}_i\}$ "very non-commutative" $\Rightarrow$ $\mathbb{E}_\varepsilon \| \sum_i \varepsilon_i \mathbf{A}_i \|_{\mathrm{op}} \lesssim \sqrt{n}$

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**Best result**

Bansal&al '23 Matrix Spencer holds if we further assume $\mathrm{rk}(\mathbf{A}_i) \lesssim n / \log^3 n$

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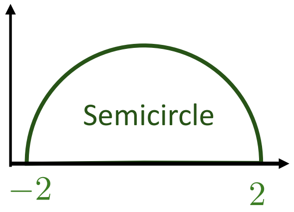

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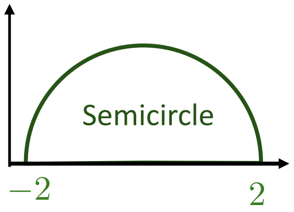

**Goals**

- Sharp **satisfiability transitions** ?
- Structure of solution space ?
- Polynomial-time solving algorithms ? [Kunisky & Zhang ’23]
- Add **structure** to $\mathbf{W}_i$ ? To probe the **Matrix Spencer conjecture** ?

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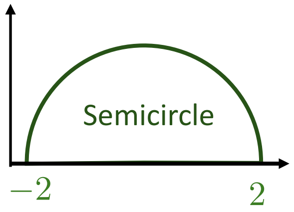

**This talk** ➢ Sharp **satisfiability transitions** ?

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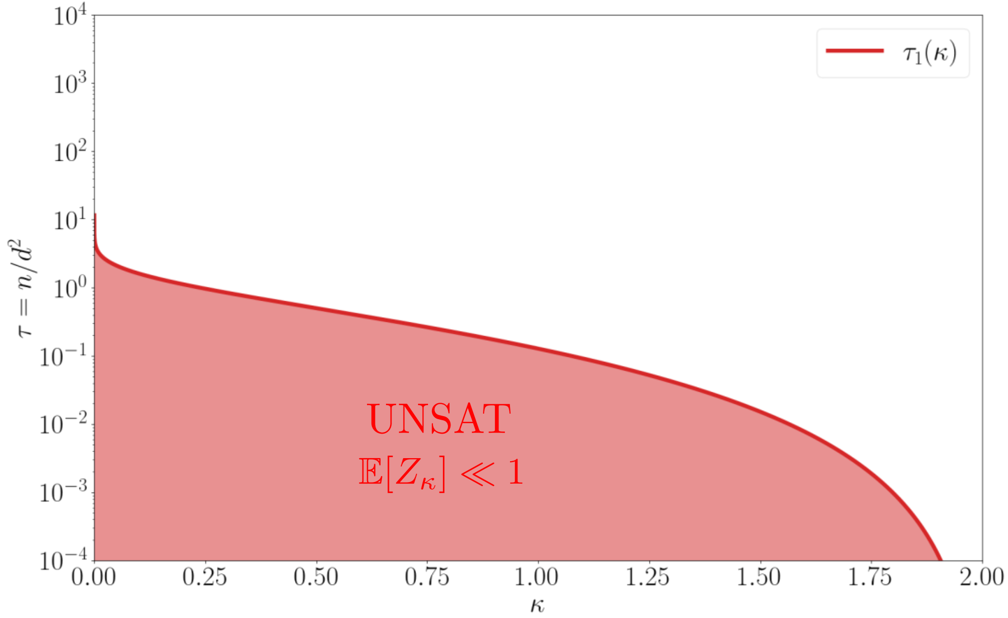

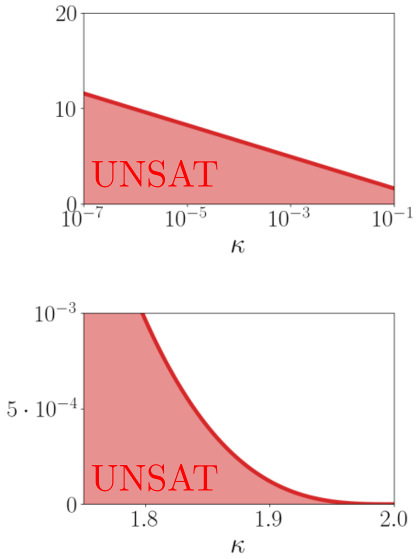

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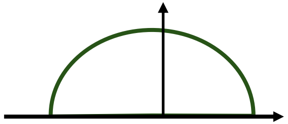
# First moment computation: a sketch



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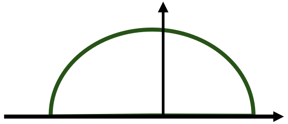

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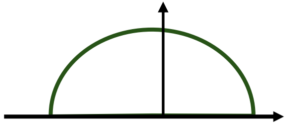

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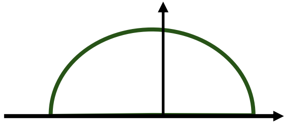

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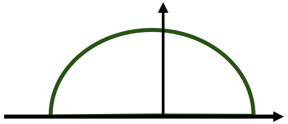

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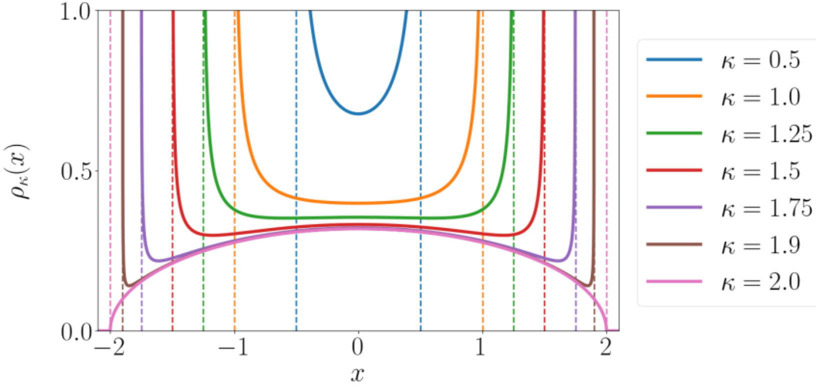

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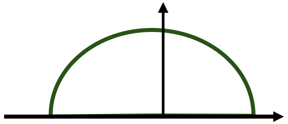

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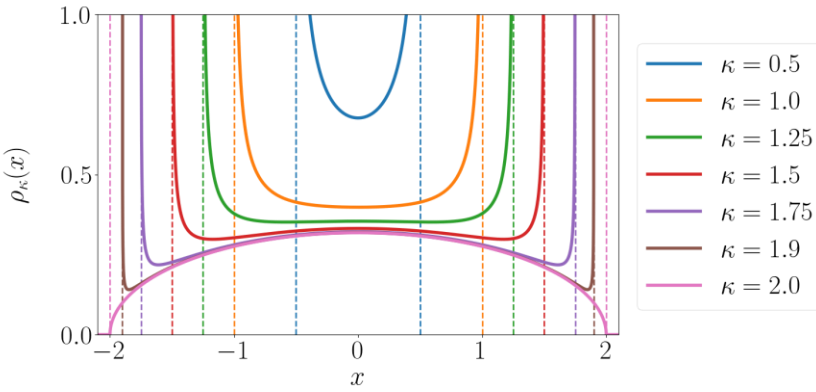

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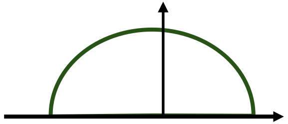

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Classical tools of logarithmic potential theory

Saff&Totik’13; Ben Arous & Guionnet ‘97

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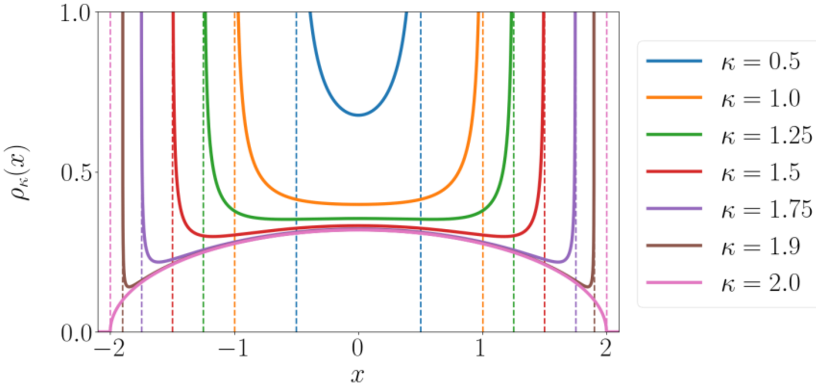

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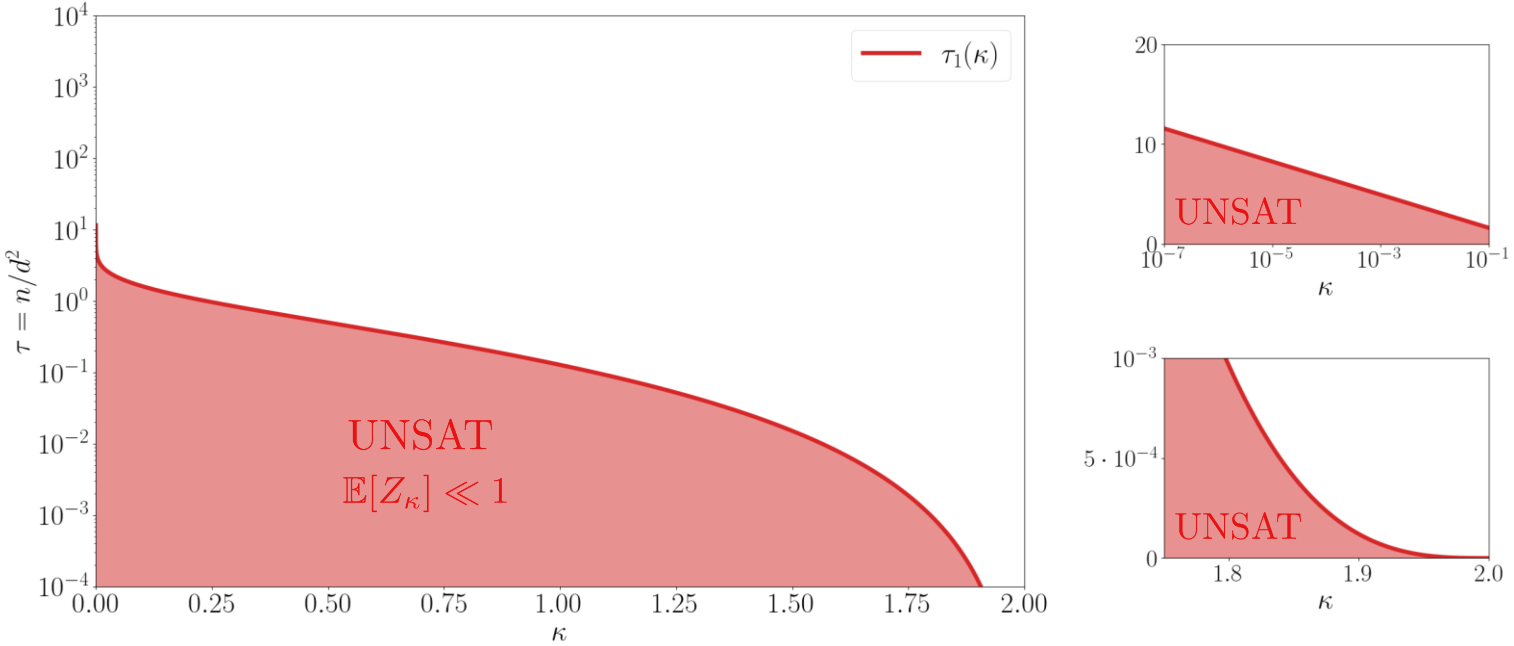
# Results II: Upper bounds via the second moment method



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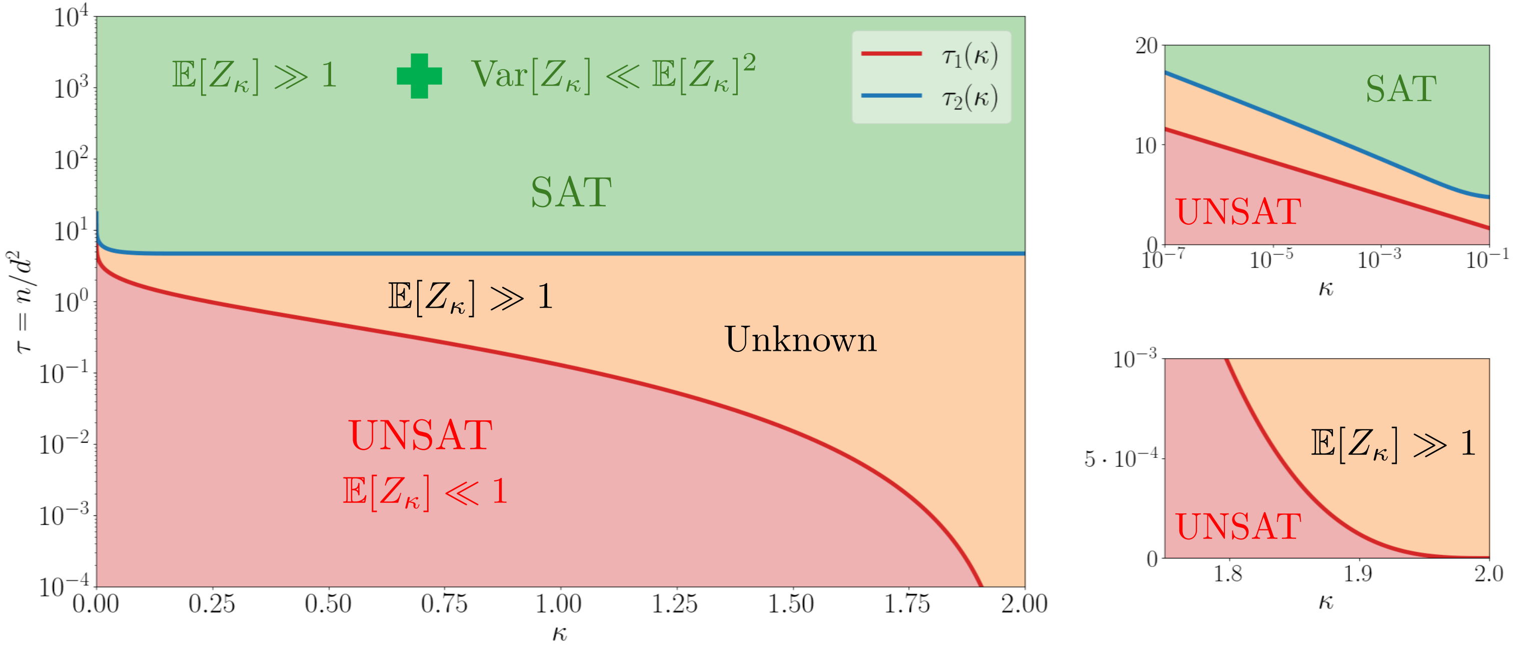

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+

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① ➕ ② $\geq \delta$

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② Second moment **upper bound** $\dfrac{\mathbb{E}[Z_\kappa^2]}{\mathbb{E}[Z_\kappa]^2} \lesssim \left[1 - \dfrac{\tau_2(\kappa)}{\tau}\right]^{-1/2}$

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$$\mathrm{disc}(\{\mathbf{W}_i\}) = \min_{\varepsilon \in \{\pm 1\}^n} \left\| \sum_{i=1}^{n} \varepsilon_i \mathbf{W}_i \right\|_{\mathrm{op}}$$

$n/d^2 \to \tau > 0$

**Theorem**

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Explicit formula (non-optimal)

① + ② $\geq \delta$

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Results of [Altschuler'23], based on Talagrand's $L^1 - L^2$ inequality

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Large deviations of the spectral norm of **correlated** $\mathrm{GOE}(d)$ matrices.

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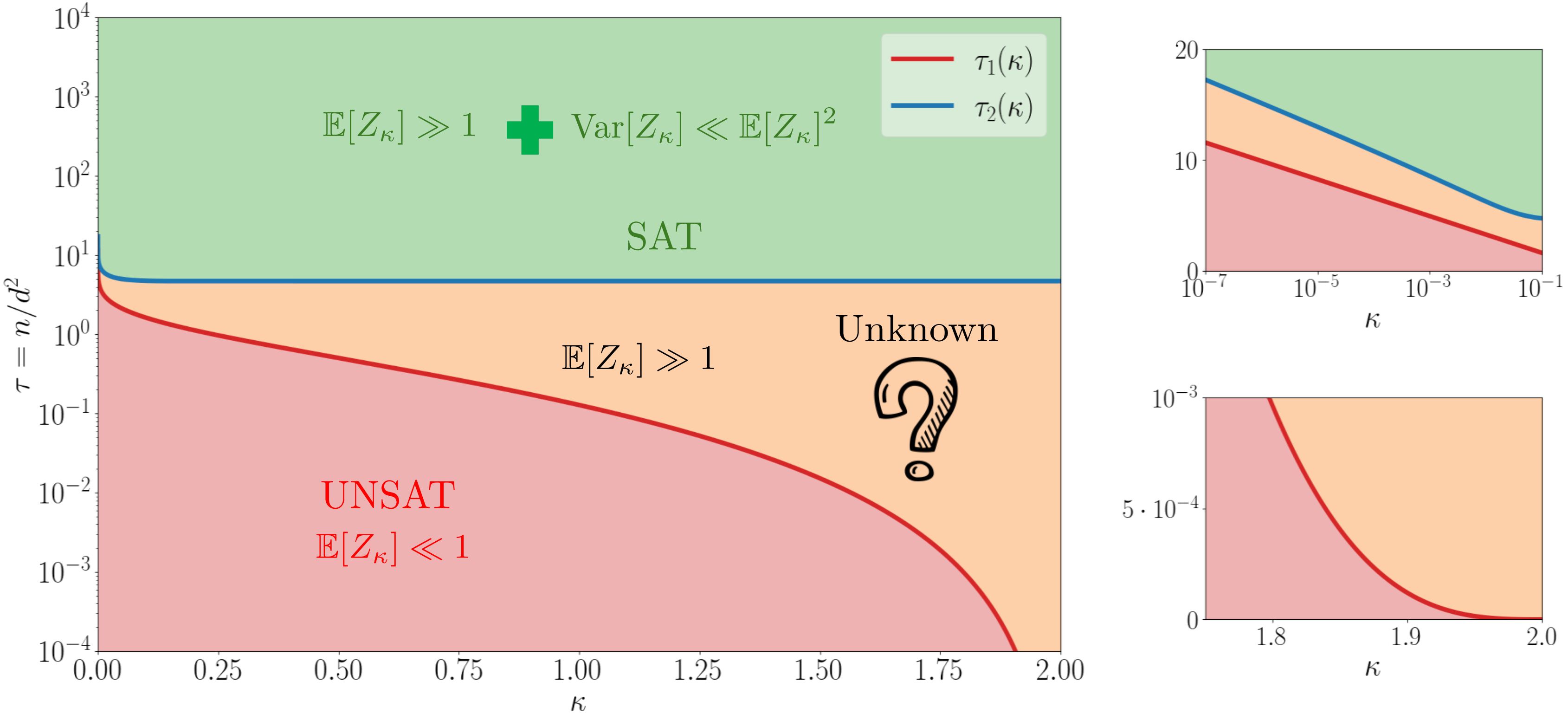
- ➢ **Discrete Laplace's method** over the overlap $q \in [-1,1]$ in $\frac{1}{2^n}\sum_{l=0}^{n}\binom{n}{l}\exp\{nG_d(q_l)\}$. ⇒ $\frac{\mathbb{E}[Z_\kappa^2]}{\mathbb{E}[Z_\kappa]^2} \lesssim \left[1 - \frac{\tau_2(\kappa)}{\tau}\right]^{-1/2}$



$\mathbb{E}[Z_\kappa] \gg 1$ ✚ $\mathrm{Var}[Z_\kappa] \ll \mathbb{E}[Z_\kappa]^2$
SAT
Unknown
?
$\mathbb{E}[Z_\kappa] \gg 1$
UNSAT
$\mathbb{E}[Z_\kappa] \ll 1$
$\tau_1(\kappa)$
$\tau_2(\kappa)$
$\tau = n/d^2$
$10^4$
$10^3$
$10^2$
$10^1$
$10^0$
$10^{-1}$
$10^{-2}$
$10^{-3}$
$10^{-4}$
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
$\kappa$
20
10
0
$10^{-7}$
$10^{-5}$
$10^{-3}$
$10^{-1}$
$\kappa$
$10^{-3}$
$5 \cdot 10^{-4}$
0
1.8
1.9
2.0
$\kappa$

# Results III: failure of the second moment method

$$G_d(q) \coloneqq \frac{1}{n} \log \frac{\mathbb{P}\left[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa \text{ and } \|q\mathbf{W} + \sqrt{1-q^2}\mathbf{Z}\|_{\mathrm{op}} \leq \kappa\right]}{\mathbb{P}[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa]^2}$$

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**Theorem**

➢ Lower bound in Laplace's method: $n/d^2 \to \tau < \tau_{\text{fail.}}(\kappa) \Rightarrow \liminf_{d\to\infty} \frac{1}{d^2} \log \frac{\mathbb{E}[Z_\kappa^2]}{\mathbb{E}[Z_\kappa]^2} > 0$

**Non-concentration** of $Z_\kappa$ on $\mathbb{E}[Z_\kappa]$.

Technical assumption: uniform continuity of $G_d''(q)$ in $q = 0$ as $d \to \infty$

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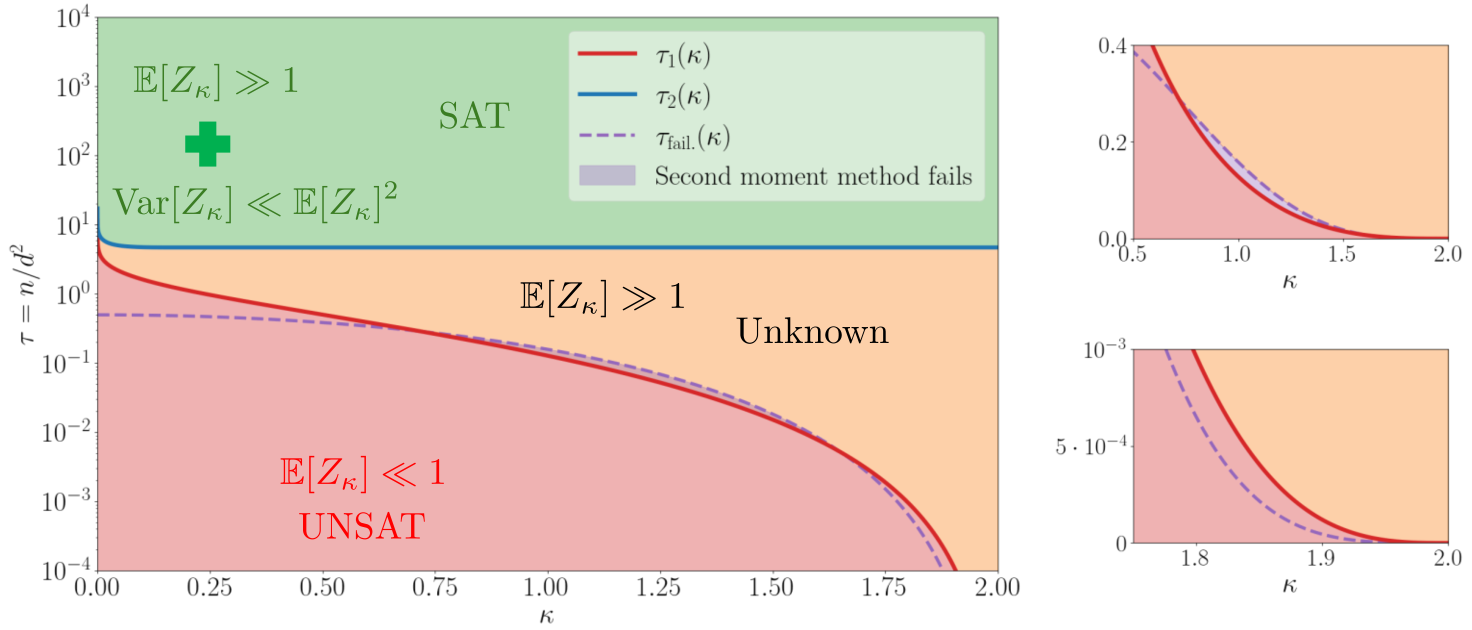


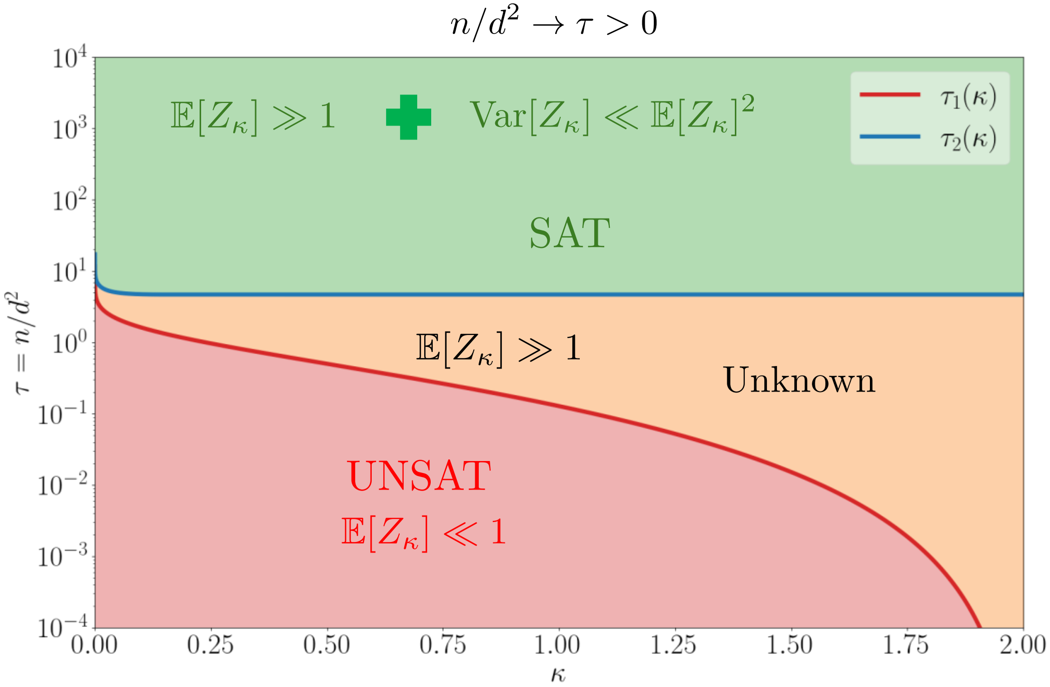
- **Purple region**: $Z_\kappa$ provably does not concentrate on its average ✚ $\mathbb{E}[Z_\kappa] \gg 1$

  Quenched $\neq$ annealed **at least** in the purple region

- The phase diagram is **more complex** than in the Symmetric Binary Perceptron !

# Summary: average-case matrix discrepancy

$$Z_\kappa := \# \left\{ \varepsilon \in \{\pm 1\}^n \text{ s.t. } \left\| \sum_{i=1}^n \varepsilon_i \mathbf{W}_i \right\|_{\mathrm{op}} \leq \kappa \sqrt{n} \right\}$$



**Matrix analog of the SBP**

✚ **Failure of the second moment method** in part of the diagram ($\neq$ Symmetric Binary Perceptron)

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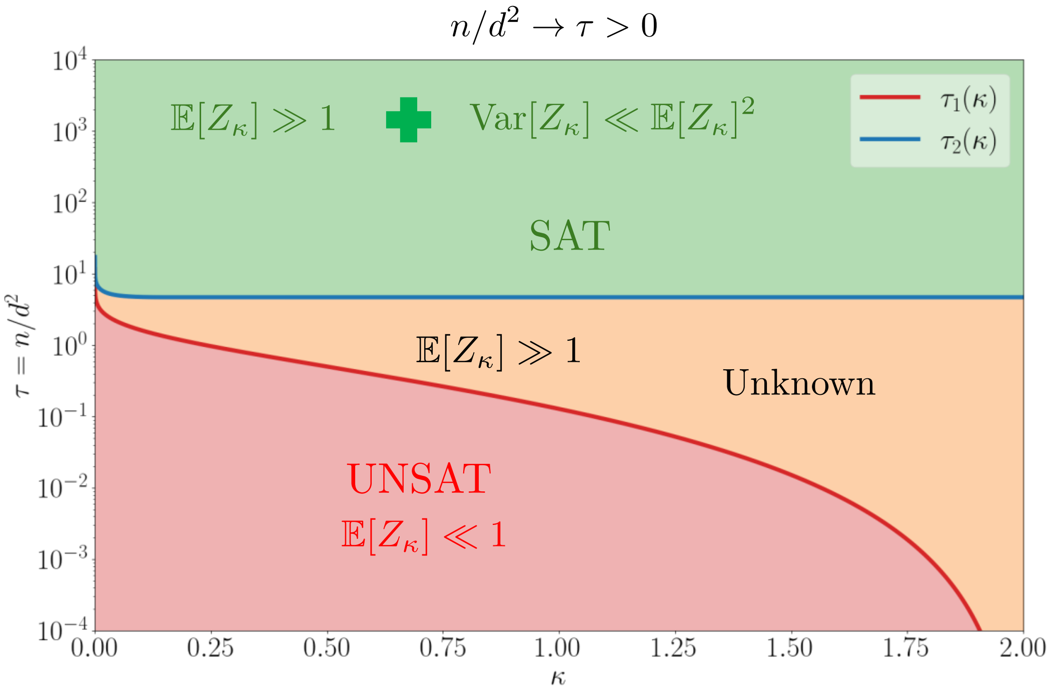

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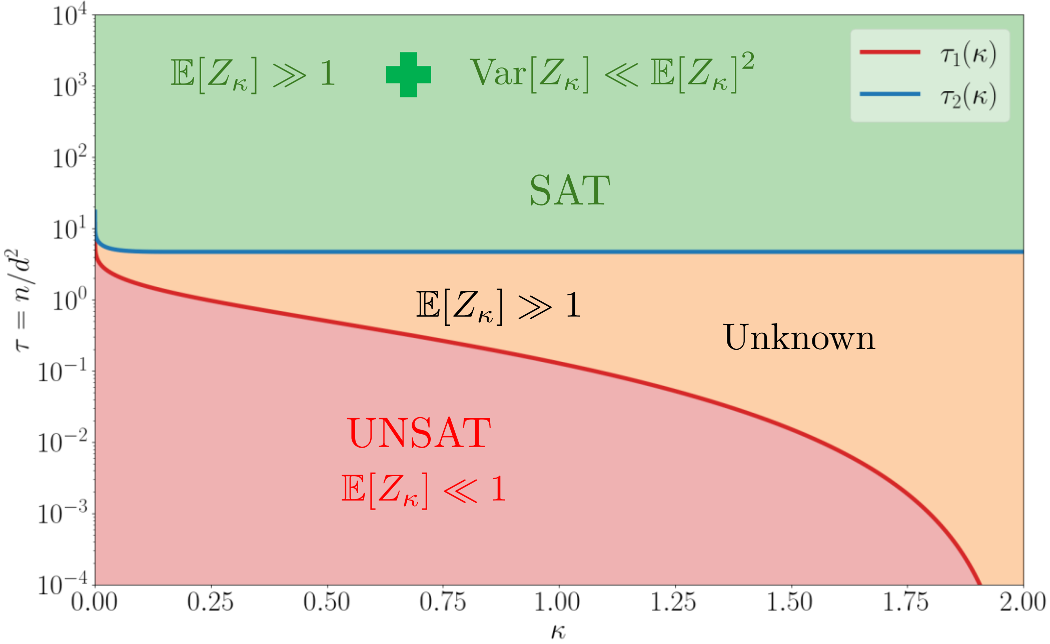

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CAUTION
WORK
IN PROGRESS

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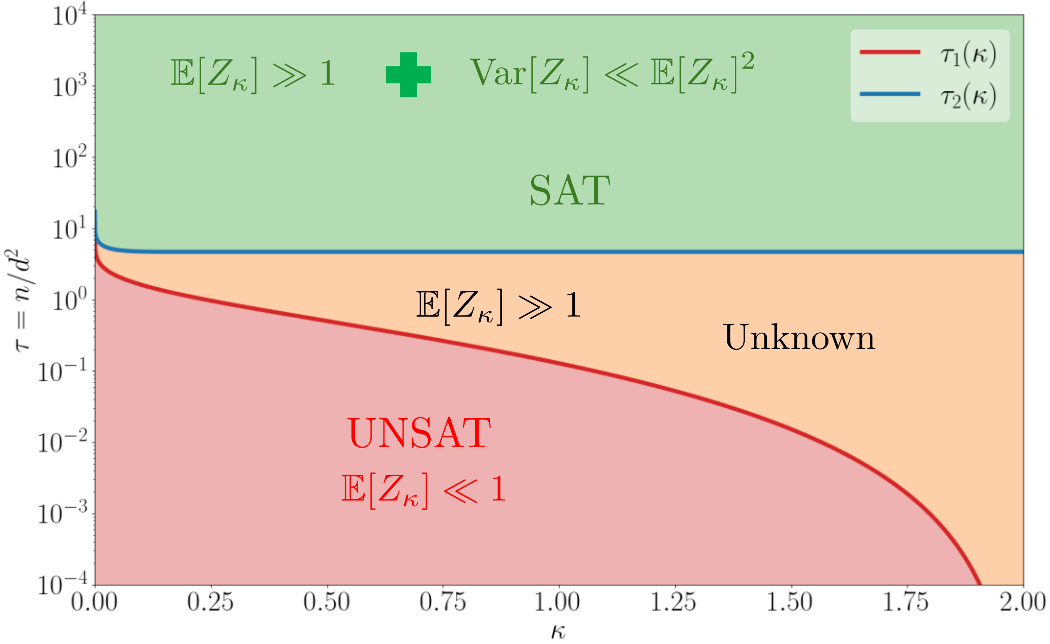

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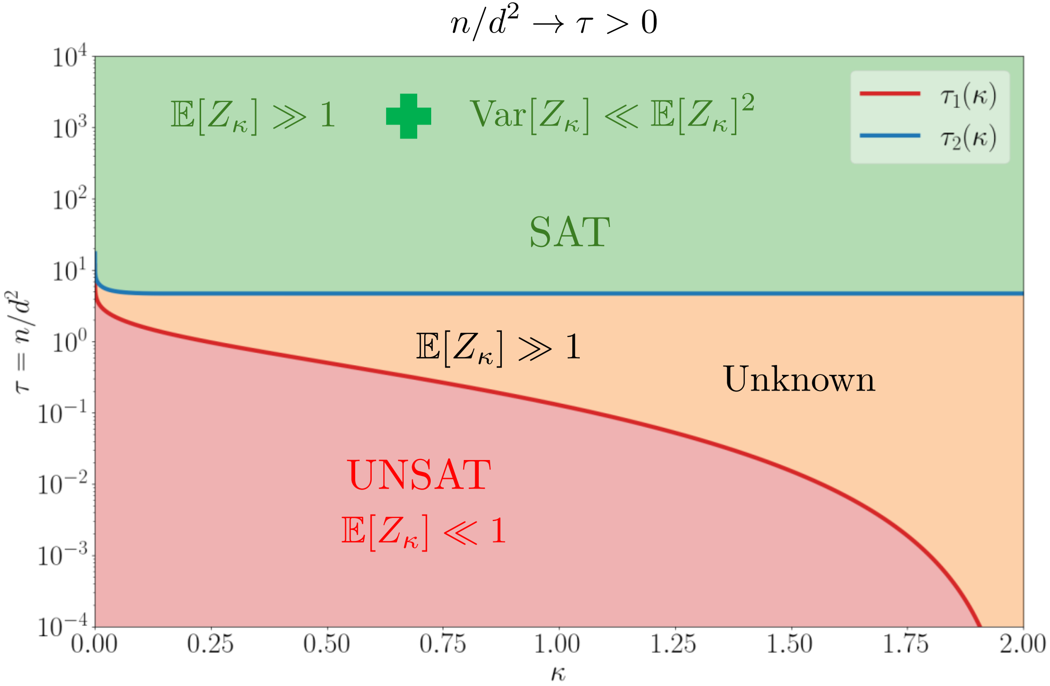

## THANK YOU !

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